Interaction of a weakly nonlinear laser pulse with a plasma

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Based on a one-dimensional model, a perturbation expansion is carried out to solve the equations describing a weakly nonlinear laser pulse in a plasma in which the electrons are treated relativistically and the plasma frequency is much less than the laser frequency. To lowest order, the expansion yields two coupled equations for the vector and scalar potentials. For a pulse which is long compared with a plasma wavelength, the coupled equations reduce to the nonlinear Schrödinger equation with well-known soliton solutions. An initial pulse of hyperbolic-secant shape which is short compared with a plasma wavelength broadens and acquires a characteristic asymmetric shape with a steep trailing edge and a much broader, gently sloping front portion, and has a frequency and wave-number shift which vary from a positive value at the front to a negative value at the rear of the pulse. The peak and rear part of a short pulse are strongly influenced by nonlinear effects, whereas the front is governed primarily by linear dispersion. The average pulse frequency continually decreases as energy is lost to the plasma wake. The wake-field phase velocity is shown to be approximately equal to the velocity of the pulse peak.

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I. INTRODUCTION

A high-power laser pulse propagating through a plasma produces a wide variety of interesting phenomena. These include plasma wake-field generation, relativistic self-focusing, frequency shifts and harmonic generation. Plasma wake fields are of importance for the understanding of beam propagation in the ionosphere, the evolution of bursts from pulsars, and for applications as high-gradient accelerators [1–4]. Even small relativistic corrections in the plasma electron motion can produce significant self-focusing of a laser beam [5]. Frequency shifts [6] and harmonic generation may have applications in plasma diagnostics or as coherent radiation sources.

In the present work, we carry out a perturbation expansion to solve the one-dimensional equations describing a weakly nonlinear laser pulse propagating in a plasma in which the electrons are treated relativistically and the ions are assumed stationary. Although the basic equations on which our analysis is based are identical to those used in the recent work by Sprangle, Esarey, and Ting [7], we do not, however, use the quasistatic approximation. Because of the use of a perturbation expansion, our results are applicable only to small-amplitude laser pulses. Also, transverse self-focusing is not included because the model is one dimensional. In spite of these restrictions, several interesting results are obtained.

It is shown that the equations governing the envelope of the laser pulse have soliton solutions provided the pulse length is long compared with a plasma wavelength and the initial pulse exceeds a certain threshold value. Even though relativistic effects on the electrons are small, they are nevertheless significant because it is shown that a model which treats the electrons nonrelativistically does not yield solitons.

For a pulse which is not long compared with a plasma wavelength, numerical solutions show that the envelope of a symmetric initial pulse typically broadens and evolves into a characteristic asymmetrical shape with a gradually sloping front portion and a relatively steep rear portion. A nonlinear frequency shift is produced, which varies spatially over the pulse. It is shown that the average pulse frequency continually decreases as the pulse loses energy to the wake. The relationship between the wake-field phase velocity and the pulse velocity is considered.

The accuracy of the quasistatic approximation [7] as applied to the weakly nonlinear laser pulse is assessed. It is found that, in the perturbation expansion, the approximation gives correct results through the two lowest orders, but that the higher-order terms are in error. Thus, for example, the strength of the third-harmonic radiation is not given correctly if the quasistatic approximation is used.

II. PERTURBATION EXPANSION

We consider the one-dimensional case in which all field and plasma variables depend on time t and on a single coordinate z. The basic equation for a linearly polarized laser pulse interacting with a cold relativistic plasma can be written in the form

$$A_{ZZ}-A_{TT}=\Omega_p^2 nA/\gamma \ , \tag{1}$$

$$\phi_{ZZ}=\Omega_p^2(n-1) \ , \tag{2}$$

$$n_T+(nu)_Z=0 \ , \tag{3}$$

$$(\gamma u)_T=\phi_Z-\gamma_Z \ , \tag{4}$$

$$\gamma=[(1+A^2)/(1-u^2)]^{1/2} \ , \tag{5}$$

where the subscripts denote partial differentiation. In these equations, A is the vector potential normalized to m_0c/e (in SI units), and ϕ is the scalar potential normalized to m_0c^2/e, where m_0, e, and c are the electron rest mass and charge and velocity of light, respectively. The electron density normalized to the unperturbed electron density is denoted by n, and the longitudinal electron fluid velocity normalized to c is denoted by u. The transverse electron fluid velocity normalized to c is given by $u_\perp = A/\gamma$. The ions are assumed to be stationary. The normalized spatial coordinate and time are defined by

$$Z = k_0 z \ , \quad T = k_0 ct \ , \tag{6}$$

where z is the spatial coordinate, t is the time, and k_0 the fundamental laser wave number. The linear dispersion relation is

$$\omega^2 = \omega_p^2 + k^2c^2 \ , \tag{7}$$

where $\omega_p = (n_0 e^2/\epsilon_0 m_0)^{1/2}$ is the electron plasma frequency, and n_0 and ϵ_0 are the unperturbed electron density and permittivity of a vacuum, respectively. The fundamental laser frequency ω_0 and wave number k_0 in the linear approximation are therefore related by $\omega_0^2 = \omega_p^2 + k_0^2 c^2$. In the subsequent analysis, we assume that

$$\omega_p / k_0 c \ll 1 \ . \tag{8}$$

Therefore, it follows from Eqs. (7) and (8) that $\omega_0 \approx k_0 c$ and $\omega_p/\omega_0 \ll 1$, i.e., the plasma is very underdense, which is usually the case in laser-plasma interactions [7–9]. It is convenient to define the normalized wave number, frequency, and plasma frequency as

$$K = k/k_0 \ , \quad \Omega = \omega / k_0 c \ , \quad \Omega_p = \omega_p / k_0 c \ , \tag{9}$$

whereby the dispersion relation (7) becomes

$$\Omega^2 = K^2 + \Omega_p^2 \ . \tag{10}$$

From (8) and (9), it is evident that $\Omega_p^2 \ll 1$. Formally, as suggested in Ref. [8], we take Ω_p to be of order ϵ, where ϵ is the expansion parameter on which the perturbation theoretic solution of Eqs. (1)–(5) derived subsequently is based. Thus, for the purposes of this formal expansion procedure, we make the replacement $\Omega_p \to \epsilon\Omega_p$, wherever Ω_p appears in the equations. It may be verified that, except for different normalization, Eqs. (1)–(5) are completely equivalent to Eqs. (1) and (2) of Sprangle, Esarey, and Ting [7].

In order to derive the solution of Eqs. (1)–(5) for case of weakly nonlinear laser pulse, the reductive perturbation method is used [10,11]. It is assumed that to lowest order, the amplitude of the vector potential is of order ϵ, which is the formal expansion parameter. The expansions

$$A = \sum_{m=1}^{\infty} \epsilon^m A^{(m)} \ , \tag{11}$$

$$n = 1 + \sum_{m=1}^{\infty} \epsilon^m n^{(m)} \ , \tag{12}$$

$$u = \sum_{m=1}^{\infty} \epsilon^m u^{(m)} \ , \tag{13}$$

$$\phi = \sum_{m=1}^{\infty} \epsilon^m \phi^{(m)} \ , \tag{14}$$

are assumed, where $A^{(m)}$, $n^{(m)}$, $u^{(m)}$, and $\phi^{(m)}$ are Fourier series given, for example, by

$$A^{(m)} = \sum_{l=-\infty}^{\infty} A_l^{(m)}(\xi,\tau)\exp(il\theta) \ , \tag{15}$$

where $A_l^{(m)} = A_{-l}^{(m)*}$ because $A^{(m)}$ is real. The quantities $n^{(m)}$, $u^{(m)}$, and $\phi^{(m)}$ are given by similar series. The slow variables ξ and τ are given by

$$\xi = \epsilon(Z - \Omega_0' T) \ , \tag{16}$$

$$\tau = \epsilon^4 T \ , \tag{17}$$

where Ω_0' is the normalized group velocity $\partial\Omega/\partial K$ evaluated at the normalized fundamental wave number $K_0 = 1$. The variable ξ is proportional to the spatial coordinate in the frame moving with the linear group velocity. The fast variable θ is given by

$$\theta = K_0 Z - \Omega_0 T = Z - \Omega_0 T \ , \tag{18}$$

where $K_0 = 1$ because of the normalization defined in (9), and $\Omega_0 = \Omega(K_0) = \Omega(1)$. The variable θ is proportional to the spatial coordinate in a frame moving with the linear phase velocity. The formulation given by Eqs. (11)–(18) is standard for the reductive perturbation method [10,11] except that the usual definition of $\tau(\tau = \epsilon^2 T)$ is replaced by Eq. (17) because of the smallness of the normalized plasma frequency as explained in Appendix A. From Eqs. (16) and (17), one obtains

$$\frac{\partial}{\partial T} = -\Omega_0 \frac{\partial}{\partial\theta} - \epsilon\Omega_0' \frac{\partial}{\partial\xi} + \epsilon^4 \frac{\partial}{\partial\tau} \ , \tag{19}$$

$$\frac{\partial}{\partial Z} = \frac{\partial}{\partial\theta} + \epsilon \frac{\partial}{\partial\xi} \ . \tag{20}$$

Using Eqs. (19) and (20), Eqs. (1)–(4) become

$$-\Omega_p^2 A_{\theta\theta} + \epsilon^2(\Omega_p/\Omega_0)^2 A_{\xi\xi} + \epsilon^2 2\Omega_0 A_{\theta\tau} + \epsilon^3 2\Omega_0' A_{\xi\tau} - \epsilon^6 A_{\tau\tau} = \Omega_p^2 (nA/\gamma) \ , \tag{21}$$

$$\phi_{\theta\theta} + \epsilon 2\phi_{\theta\xi} + \epsilon^2\phi_{\xi\xi} = \epsilon^2\Omega_p^2(n-1) \ , \tag{22}$$

$$[n(\Omega_0 - u)]_\theta + \epsilon[n(\Omega_0' - u)]_\xi = \epsilon^4 n_\tau \ , \tag{23}$$

$$[\gamma(1 - \Omega_0 u) - \phi]_\theta + \epsilon[\gamma(1 - \Omega_0' u) - \phi]_\xi = -\epsilon^4(\gamma u)_\tau \ , \tag{24}$$

where Eq. (10) has been used, and the replacement $\Omega_p \to \epsilon\Omega_p$ has been carried out in Eqs. (1), (2), and (10). In Eqs. (21)–(24), the quantities Ω_0 and Ω_0' are obtained from Eq. (10) (with $\Omega_p \to \epsilon\Omega_p$) as

$$\Omega_0 = \Omega(K=1) = (1 + \epsilon^2\Omega_p^2)^{1/2} = 1 + \tfrac{1}{2}\epsilon^2\Omega_p^2 - \tfrac{1}{8}\epsilon^4\Omega_p^4 + \cdots \ , \tag{25}$$

$$\Omega_0' = \left[\frac{\partial\Omega}{\partial K}\right]_{K=1} = \Omega_0^{-1} = 1 - \tfrac{1}{2}\epsilon^2\Omega_p^2 + \tfrac{3}{8}\epsilon^4\Omega_p^4 + \cdots \ . \tag{26}$$

At order ϵ, it is assumed that $A^{(1)} \neq 0$ but that

$$n^{(1)} = u^{(1)} = \phi^{(1)} = 0 \ , \tag{27}$$

because the perturbations in electron density, longitudinal electron velocity, and scalar potential are generated by $A^{(1)}$ through the nonlinear terms in Eqs. (1)–(4) and are therefore of order higher than first. Hence, with $u^{(1)}=0$, Eqs. (5), (11), and (13) yield

$$\gamma=1+\tfrac{1}{2}\epsilon^2(A^{(1)})^2+\epsilon^3A^{(1)}A^{(2)}+\epsilon^4[A^{(1)}A^{(3)}+\tfrac{1}{2}(A^{(2)})^2+\tfrac{1}{2}(u^{(2)})^2-\tfrac{1}{8}(A^{(1)})^4] +\epsilon^5[A^{(1)}A^{(4)}+A^{(2)}A^{(3)}+u^{(2)}u^{(3)}-\tfrac{1}{2}(A^{(1)})^3A^{(2)}]+\cdots , \quad (28)$$

and with $n^{(1)}=u^{(1)}=0$, Eqs. (5), (11), (12), and (13) yield

$$\frac{n}{\gamma}=1+\epsilon^2[-\tfrac{1}{2}(A^{(1)})^2+n^{(2)}] +\epsilon^4[-A^{(1)}A^{(3)}-\tfrac{1}{2}(A^{(2)})^2+\tfrac{3}{8}(A^{(1)})^4 -\tfrac{1}{2}(u^{(2)})^2-\tfrac{1}{2}(A^{(1)})^2n^{(2)}+n^{(4)}]+\cdots . \quad (29)$$

The procedure is to insert Eqs. (11)–(14) and (25)–(29) into Eqs. (21)–(24) and collect terms of the same order in ϵ. Since ϵ only serves as a bookkeeping parameter, it is set equal to unity after the procedure is completed.

A. Order ϵ

At order ϵ, Eqs. (22)–(24) are satisfied because of (27), and (21) gives $A^{(1)}_{\theta\theta}=-A^{(1)}$, which, using Eq. (15) yields $A^{(1)}_0=0$ and

$$A^{(1)}=A^{(1)}_{-1}\exp(-i\theta)+A^{(1)}_1\exp(i\theta) , \quad (30)$$

where $A^{(1)}_{\pm1}$ are functions of ξ and τ to be determined.

B. Order ϵ^2

At order ϵ^2, Eq. (21) gives $A^{(2)}_{\theta\theta}=-A^{(2)}$, which yields $A^{(2)}_0=A^{(2)}_2=0$ and

$$A^{(2)}=A^{(2)}_{-1}\exp(-i\theta)+A^{(2)}_1\exp(-i\theta) . \quad (31)$$

Equation (22) gives $\phi^{(2)}_{\theta\theta}=0$, which yields

$$\phi^{(2)}_l=0 , \quad l\neq 0 . \quad (32)$$

Equation (23) gives $(n^{(2)}-u^{(2)})_\theta=0$, which yields

$$n^{(2)}=u^{(2)} , \quad l\neq 0 . \quad (33)$$

Equation (24) gives $[(A^{(1)})^2/2-u^{(2)}-\phi^{(2)}]_\theta=0$, which yields

$$u^{(2)}=(A^{(1)})^2/2-\phi^{(2)} , \quad l\neq 0 . \quad (34)$$

C. Order ϵ^3

At order ϵ^3, Eq. (21) gives

$$\Omega_p^2(A^{(3)}_{\theta\theta}+A^{(3)})=2A^{(1)}_{\theta\tau}+\Omega_p^2A^{(1)}_{\xi\xi} +\Omega_p^2[(A^{(1)})^2/2-n^{(2)}]A^{(1)} . \quad (35)$$

For $l=0$, 2, and 3, Eq. (35) yields

$$A^{(3)}_0=A^{(3)}_2=A^{(3)}_3=0 , \quad (36)$$

where Eqs. (32)–(34) have been used. For $l=1$, Eq. (35) gives

$$2iA^{(1)}_{1\tau}+\Omega_p^2A^{(1)}_{1\xi\xi}+\Omega_p^2[(A^{(1)})^2/2-n^{(2)}]_0A^{(1)}_1=0 , \quad (37)$$

where Eqs. (32)–(34) have been used. Equation (22) gives $\phi^{(3)}_{\theta\theta}=0$, where Eq. (32) has been used. Thus

$$\phi^{(3)}_l=0 , \quad l\neq 0 . \quad (38)$$

Equation (23) gives

$$(n^{(3)}-u^{(3)})_\theta+(n^{(2)}-u^{(2)})_\xi=0 . \quad (39)$$

For $l=0$, Eq. (39) yields $n^{(2)}_0=u^{(2)}_0$, which, together with Eq. (33), shows that

$$n^{(2)}=u^{(2)} . \quad (40)$$

For $l\neq 0$, Eqs. (39) and (40) yield

$$n^{(3)}=u^{(3)} , \quad l\neq 0 . \quad (41)$$

Equation (24) gives

$$(A^{(1)}A^{(2)}-u^{(3)}-\phi^{(3)})_\theta+[(A^{(1)})^2/2-u^{(2)}-\phi^{(2)}]_\xi=0 . \quad (42)$$

For $l=0$, this yields $u^{(2)}_0=[(A^{(1)})^2]_0/2-\phi^{(2)}_0$, which, with Eqs. (34) and (40), shows that

$$n^{(2)}=u^{(2)}=(A^{(1)})^2/2-\phi^{(2)}_0 . \quad (43)$$

For $l\neq 0$, Eq. (42), together with (38) and (41), yields

$$n^{(3)}=u^{(3)}=A^{(1)}A^{(2)} , \quad l\neq 0 . \quad (44)$$

Inserting Eq. (43) into (37), one obtains

$$iA^{(1)}_{1\tau}+\tfrac{1}{2}\Omega_p^2A^{(1)}_{1\xi\xi}+\tfrac{1}{2}\Omega_p^2\phi^{(2)}_0A^{(1)}_1=0 . \quad (45)$$

D. Order ϵ^4

At order ϵ^4, Eq. (21) gives

$$2A^{(2)}_{\theta\tau}+\Omega_p^2A^{(2)}_{\xi\xi}+\Omega_p^2\phi^{(2)}_0A^{(2)} =\Omega_p^2(A^{(4)}_{\theta\theta}+A^{(4)})-2A^{(1)}_{\xi\tau}-\Omega_p^2(A^{(1)}A^{(2)}-n^{(3)})A^{(1)} , \quad (46)$$

where Eq. (43) has been used. For $l\neq 1$, Eq. (46), together with Eqs. (30), (31), (36), and (44), yields

$$A^{(4)}_l=0 , \quad l\neq 1 . \quad (47)$$

For $l=1$, Eq. (46) gives

$$2iA^{(2)}_{1\tau}+\Omega_p^2A^{(2)}_{1\xi\xi}+\Omega_p^2\phi^{(2)}_0A^{(2)}_1 =-2A^{(1)}_{1\xi\tau}-\Omega_p^2(A^{(1)}A^{(2)}-n^{(3)})_0A^{(1)}_1 , \quad (48)$$

where Eq. (44) has been used. Equation (22) gives

$$\phi^{(4)}_{\theta\theta}+\phi^{(2)}_{\xi\xi}+\Omega_p^2\phi^{(2)}=\Omega_p^2(A^{(1)})^2/2 , \quad (49)$$

where Eq. (38) has been used. For $l=0$, Eq. (49) yields

$$\phi^{(2)}_{0\xi\xi}+\Omega_p^2\phi^{(2)}_0=\Omega_p^2|A^{(1)}_1|^2 \ , \quad (50)$$

where Eq. (30) has been used together with $A^{(1)}_1=A^{(1)*}_{-1}$. For $l=1$, 3, and 4, Eq. (49), together with Eqs. (30) and (32), yields

$$\phi^{(4)}_1=\phi^{(4)}_3=\phi^{(4)}_4=0 \ , \quad (51)$$

whereas for $l=2$, one obtains

$$\phi^{(4)}_2=-\Omega_p^2(A^{(1)}_1)^2/8 \ . \quad (52)$$

Equation (23) gives

$$(n^{(4)}+\Omega_p^2 n^{(2)}/2-u^{(4)}-n^{(2)}u^{(2)})_\theta+(n^{(3)}-u^{(3)})_\xi=0 \ . \quad (53)$$

For $l=0$, Eq. (53) yields $n^{(3)}_0=u^{(3)}_0$, which, with Eq. (41), shows that

$$n^{(3)}=u^{(3)} \ . \quad (54)$$

For $l\neq0$, Eq. (53) gives

$$n^{(4)}_l=u^{(4)}_l-\Omega_p^2 n^{(2)}_l/2+[(n^{(2)})^2]_l \ , \quad l\neq0 \ , \quad (55)$$

where $n^{(2)}$ is given by Eq. (43). Equation (24) gives

$$\{A^{(1)}A^{(3)}+\tfrac{1}{2}(A^{(2)})^2+\tfrac{1}{2}u^{(2)}[u^{(2)}-(A^{(1)})^2-\Omega_p^2] -\tfrac{1}{8}(A^{(1)})^4-u^{(4)}-\phi^{(4)}\}_\theta +(A^{(1)}A^{(2)}-u^{(3)}-\phi^{(3)})_\xi=0 \ . \quad (56)$$

For $l=0$, Eq. (56) yields $u^{(3)}_0=(A^{(1)}A^{(2)})_0-\phi^{(3)}_0$, which together with Eqs. (38), (44), and (54) shows that

$$n^{(3)}=u^{(3)}=A^{(1)}A^{(2)}-\phi^{(3)}_0 \ . \quad (57)$$

For $l\neq0$, Eq. (56), together with (40) and (57), yields

$$u^{(4)}_l=[A^{(1)}A^{(3)}+\tfrac{1}{2}(A^{(2)})^2+\tfrac{1}{2}(n^{(2)})^2 -\tfrac{1}{8}(A^{(1)})^4-\tfrac{1}{2}(A^{(1)})^2 n^{(2)} -\tfrac{1}{2}\Omega_p^2 n^{(2)}-\phi^{(4)}]_l \ , \quad l\neq0 \ . \quad (58)$$

From Eqs. (55) and (58), it can be shown that

$$n^{(4)}_1=u^{(4)}_1=n^{(4)}_3=u^{(4)}_3=n^{(4)}_4=0 \ , \quad u^{(4)}_4=-(A^{(1)}_1)^2/4 \ , \quad (59)$$

$$u^{(4)}_2=n^{(4)}_2+(A^{(1)}_1)^2\phi^{(2)}_0+\tfrac{1}{4}(A^{(1)}_1)^2(\Omega_p^2-4A^{(1)}_{-1}A^{(1)}_1) =A^{(1)}_1A^{(3)}_1+\tfrac{1}{2}(A^{(2)}_1)^2-\tfrac{1}{8}\Omega_p^2(A^{(1)}_1)^2 -(A^{(1)}_1)^3A^{(1)}_{-1} \ . \quad (60)$$

Inserting Eq. (57) into (48), one obtains

$$iA^{(2)}_{1\tau}+\tfrac{1}{2}\Omega_p^2 A^{(2)}_{1\xi\xi}+\tfrac{1}{2}\Omega_p^2\phi^{(2)}_0A^{(2)}_1 =-A^{(1)}_{1\xi\tau}-\tfrac{1}{2}\Omega_p^2\phi^{(3)}_0A^{(1)}_1 \ . \quad (61)$$

E. Order ϵ^5

At order ϵ^5, Eq. (21) gives

$$2A^{(3)}_{\theta\tau}+\Omega_p^2A^{(3)}_{\xi\xi}+\Omega_p^2\phi^{(2)}_0A^{(3)}=\Omega_p^2(A^{(5)}_{\theta\theta}+A^{(5)})-2A^{(2)}_{\xi\tau}-\Omega_p^2(A^{(1)}_{\theta\tau}-\Omega_p^2A^{(1)}_{\xi\xi}) -\Omega_p^2\phi^{(3)}_0A^{(2)}+\Omega_p^2A^{(1)}[n^{(4)}-A^{(1)}A^{(3)}-\tfrac{1}{2}(A^{(2)})^2+\phi^{(2)}_0(A^{(1)})^2-\tfrac{1}{2}(\phi^{(2)}_0)^2] \ , \quad (62)$$

where Eqs. (43) and (57) have been used. With the use of Eqs. (30), (31), (36), (59), and (60), Eq. (62) yields

$$A^{(5)}_0=A^{(5)}_2=A^{(5)}_4=A^{(5)}_5=0 \ , \quad (63)$$

$$A^{(5)}_3=-3\Omega_p^2(A^{(1)}_1)^3/64 \ . \quad (64)$$

Equation (22) gives

$$\phi^{(5)}_{\theta\theta}+2\phi^{(4)}_{\theta\xi}+\phi^{(3)}_{0\xi\xi}+\Omega_p^2\phi^{(3)}_0=\Omega_p^2A^{(1)}A^{(2)} \ , \quad (65)$$

where Eqs. (38) and (57) have been used. For $l=0$, Eq. (65) yields

$$\phi^{(3)}_{0\xi\xi}+\Omega_p^2\phi^{(3)}_0=\Omega_p^2(A^{(1)}_{-1}A^{(2)}_1+A^{(1)}_1A^{(2)}_{-1}) \ , \quad (66)$$

where Eqs. (30) and (31) have been used. Using Eqs. (51) and (52), Eq. (65) yields

$$\phi^{(5)}_1=\phi^{(5)}_3=\phi^{(5)}_4=\phi^{(5)}_5=0 \ , \quad (67)$$

$$\phi^{(5)}_2=-\Omega_p^2\{i[(A^{(1)}_1)^2]_\xi+2A^{(1)}_1A^{(2)}_1\}/8 \ . \quad (68)$$

Equation (23) gives

$$[n^{(2)}u^{(3)}-n^{(3)}(\tfrac{1}{2}\Omega_p^2-u^{(2)})-n^{(5)}+u^{(5)}]_\theta -[n^{(4)}-u^{(4)}-n^{(2)}(\tfrac{1}{2}\Omega_p^2+u^{(2)})]_\xi=0 \ . \quad (69)$$

For $l=0$, Eq. (69) yields

$$n^{(4)}_0=u^{(4)}_0+\Omega_p^2n^{(2)}_0/2+[(n^{(2)})^2]_0 \ ,$$

which, with Eq. (55), shows that

$$n^{(4)}=u^{(4)}+(n^{(2)})^2-\Omega_p^2(1-2\delta_{0l})n^{(2)}/2 \ , \quad (70)$$

where δ_{0l} is the Kronecker delta which is unity when $l=0$ and vanishes for $l\neq0$. The complete Eq. (24) at order ϵ^5 is not given here because of its length, but the $l=0$ part is

$$u^{(4)}_0=[A^{(1)}A^{(3)}+\tfrac{1}{2}(A^{(2)})^2+\tfrac{1}{2}(n^{(2)})^2 -\tfrac{1}{8}(A^{(1)})^4-\tfrac{1}{2}(A^{(1)})^2n^{(2)}+\tfrac{1}{2}\Omega_p^2n^{(2)}-\phi^{(4)}]_0 \ ,$$

which, with Eq. (58), shows that

$$u^{(4)}=A^{(1)}A^{(3)}+\tfrac{1}{2}(A^{(2)})^2+\tfrac{1}{2}(n^{(2)})^2 -\tfrac{1}{8}(A^{(1)})^4-\tfrac{1}{2}(A^{(1)})^2n^{(2)}-\phi^{(4)} -\tfrac{1}{2}\Omega_p^2(1-2\delta_{0l})n^{(2)} . \quad (71)$$

In summary, the reductive perturbation method through order ϵ^5 has been applied to Eqs. (21)–(24), which results in a hierarchy of equations. The first level of the hierarchy is comprised of Eqs. (45) and (50), which are two coupled equations for $A_1^{(1)}$ and $\phi_0^{(2)}$, and Eq. (43), which gives $n^{(2)}$ and $u^{(2)}$ in terms of $A_1^{(1)}$ and $\phi_0^{(2)}$. Solutions of Eqs. (45) and (50) are presented in Secs. III and IV. The second level in the hierarchy is comprised of the coupled Eqs. (61) and (66) for $A_1^{(2)}$ and $\phi_0^{(3)}$, together with Eq. (57), which gives $n^{(3)}$ and $u^{(3)}$ in terms of $A_1^{(1)}$, $A_1^{(2)}$, and $\phi_0^{(3)}$. Because some of the equations become increasingly lengthy with increasing hierarchy level, the reductive perturbation procedure has not been carried out completely here beyond the second level. Nevertheless, from the equations that were presented, the following interesting results regarding the harmonic content of the field and plasma variables have been derived. The vector potential has no harmonics until order ϵ^5 where the third-harmonic component $A_3^{(5)}$ appears and is given by Eq. (64). Through order ϵ^3, the scalar potential has no dependence on the fast variable θ, i.e., only the $l=0$ components $\phi_0^{(2)}$ and $\phi_0^{(3)}$ are present; at order ϵ^4 and ϵ^5 (and presumably at higher order) it also has second-harmonic ($l=2$) components $\phi_2^{(4)}$ and $\phi_2^{(5)}$ given by Eqs. (52) and (68) in addition to the $l=0$ components $\phi_0^{(4)}$ and $\phi_0^{(5)}$. The electron density and longitudinal velocity have both $l=0$ and 2 components through order ϵ^4. In addition, at order ϵ^4, the longitudinal electron velocity has a fourth-harmonic ($l=4$) component $u_4^{(4)}$ given by Eq. (59).

The foregoing results are based on the assumption that $\omega_p/\omega_0 \ll 1$. It is interesting to compare the results of a perturbation expansion based on the assumption that ω_p/ω_0 is of order unity. In that case, the normalized group dispersion Ω_0'' is of order unity, and therefore, as noted in Appendix A, the linear solution suggests that the variable τ given by Eq. (17) should be replaced by $\tau=\epsilon^2 T$. A perturbation expansion based on this scaling has been carried out by Kates and Kaup [12] which, for the present assumptions of immobile ions, cold electrons, and linear polarization, yields the lowest-order results (in the present notation)

$$iA_{1\tau}^{(1)}+\frac{\Omega_p^2}{2\Omega_0^3}A_{1\xi\xi}^{(1)} +\frac{\Omega_p^2}{2\Omega_0}\left[\frac{3}{2}-\frac{2}{(4\Omega_0^2-\Omega_p^2)}\right]|A_1^{(1)}|^2A_1^{(1)}=0 , \quad (72)$$

$$\phi_0^{(2)}=|A_1^{(1)}|^2 , \quad (73)$$

$$n_0^{(2)}=u_0^{(2)}=0 , \quad n_2^{(2)}=u_2^{(2)}=(A_1^{(1)})^2/2 , \quad (74)$$

where $\Omega_0^2=1+\Omega_p^2$ from Eq. (10) with $K_0=1$. Thus, the coupled equations (45) and (50) for $A_1^{(1)}$ and $\phi_0^{(2)}$ are replaced by the nonlinear Schrödinger equation (72) for $A_1^{(1)}$, whose solution then yields $\phi_0^{(2)}$ directly from Eq. (73). Moreover, Eq. (74) yields vanishing dc parts of $n^{(2)}$ and $u^{(2)}$, in contrast to Eq. (43), which generally gives a nonzero dc part. It is shown in Sec. III that, for a pulse which is long compared with a plasma wavelength, Eqs. (72)–(74) with $\Omega_p \ll 1$ give results which coincide with those of the expansion presented here. For a short pulse with $\Omega_p \ll 1$, however, the results of Sec. IV show that the coupled equations (45) and (50) give results which differ markedly from those of Eqs. (72) and (73). In particular, Eqs. (45) and (50) yield short-pulse properties including asymmetrical shape, frequency and wave-number shifts, and wake-field generation which are not reproduced by Eqs. (72) and (73).

III. LONG PULSE

To lowest order, the vector and scalar potentials are governed by the coupled equations (45) and (50). Based on these equations, we consider first the case in which the laser pulse has a length l which is long compared with the plasma wavelength, i.e., $\omega_p l/c \gg 1$. To justify our neglect of the ion motion, it is required, however, that the pulse duration be short compared with the ion period, i.e., $\omega_{pi} l/c \ll 1$. Some previous results for this case are summarized in Ref. [13]. In this section, we briefly indicate how the present theory yields results which are in agreement with some of the more important results of previous workers. Using $\phi_{0\xi\xi}^{(2)} \sim \phi_0^{(2)}/L^2$, where $L=k_0 l$ and l is the characteristic envelope scale length, then the ratio of the second term on the right-hand side of Eq. (50) to the first term is of order $\Omega_p^2 L^2=\omega_p^2 l^2/c^2 \gg 1$, whereby Eq. (50) gives approximately

$$\phi_0^{(2)}=|A_1^{(1)}|^2 , \quad (75)$$

and thus Eq. (45) becomes approximately

$$iA_{1\tau}^{(1)}+\tfrac{1}{2}\Omega_p^2 A_{1\xi\xi}^{(1)}+\tfrac{1}{2}\Omega_p^2|A_1^{(1)}|^2|A_1^{(1)}=0 , \quad (76)$$

which is the nonlinear Schrodinger (NLS) equation [14]. Using Eq. (75), Eq. (43) yields

$$n_0^{(2)}=u_0^{(2)}=0 , \quad n_2^{(2)}=u_2^{(2)}=(A_1^{(1)})^2/2 . \quad (77)$$

It is noted that if we assume that $\Omega_p \ll 1$ and therefore neglect Ω_p^2 compared with $\Omega_0^2 \approx 1$ in Eqs. (72)–(74), then we obtain Eqs. (75)–(77). Therefore, for the case of a long pulse, the lowest-order results of Kates and Kaup [12] are identical to those derived here. Also, Eq. (76) is equivalent to the NLS equation derived in Ref. [13] except that the coefficient of the second (dispersive) term in Ref. [13] is $\frac{1}{2}$ (in the present notation) instead of $\Omega_p^2/2$, which is appropriate for $\omega \approx \omega_p$ rather than $\omega \gg \omega_p$, as assumed here.

It is well known that the NLS equation (76) admits soliton solutions [14–16]. The simplest single-soliton solution is

$$A_1^{(1)}=A\,\mathrm{sech}(A\xi/2^{1/2})\exp(iA^2\Omega_p^2\tau/4) , \quad (78)$$

where A is a constant. Equation (78) represents a nonlinear pulse traveling at the linear group velocity Ω_0', which preserves its shape because of a balance between dispersion and nonlinearity. A similar analytic result has been obtained recently by Kaw, Sen, and Katsouleas [17] for a small-amplitude circularly polarized wave. For a large-amplitude circularly polarized wave, Kaw, Sen, and Katsouleas [17] obtain solitary-wave solutions numerically. It is of interest to compare the characteristic time scale of the soliton given by Eq. (78) with that of relevant parametric instabilities. From Eq. (78), it is evident that the characteristic time on which the soliton varies is $\tau_s \sim 4A^{-2}\Omega_p^{-2}$. On the other hand, for a long pulse, the characteristic time on which stimulated Raman backscattering occurs is approximately that obtained for unbounded plane waves, given (in normalized units) by [18] $\tau_R \sim A^{-1}\Omega_p^{-1/2}/2$. Thus, $\tau_R/\tau_s \sim A\Omega_p^{3/2}/8$, which is small compared with unity because $A \ll 1$ and $\Omega_p \ll 1$. Therefore, it follows that the long-pulse soliton given by Eq. (78) would probably be difficult to observe experimentally because the instability due to stimulated Raman backscattering occurs on a shorter time scale than the soliton time scale.

IV. ARBITRARY-LENGTH PULSE

For a pulse which is not long compared with a plasma wavelength, it is necessary to solve Eqs. (45) and (50) numerically. It is noted that Eqs. (45) and (50) are invariant under the transformations

$$A_1^{(1)} = \epsilon A_1^{(1)'} , \quad \phi_0^{(2)} = \epsilon^2 \phi_0^{(2)'} ,$$
$$\xi = \frac{\xi'}{\epsilon} , \quad \tau = \frac{\tau'}{\epsilon^4} , \quad \Omega_p = \epsilon \Omega_p' , \tag{79}$$

i.e., the unprimed variables in Eqs. (45) and (50) can be replaced by the primed variables defined by (79). Equations (45) and (50) in the primed variables may be solved with Ω_p' of order unity and an initial function $A_1^{(1)'}(\xi',\tau'=0)$ whose amplitude is of order unity and whose dependence on the spatial coordinate ξ' has a scale length of order unity. The resulting solutions $A_1^{(1)'}(\xi',\tau')$ and $\phi_0^{(2)'}(\xi',\tau')$ will typically have amplitudes of order unity, and will also have temporal and spatial scales of order unity. These solutions can then be scaled by means of the transformations (79) to include the smallness of $A_1^{(1)}$, $\phi_0^{(2)}$, and Ω_p, and the long scales of τ and ξ. It is important to note that the parameter $\Omega_p L = \omega_p l/c = k_p l$, which determines the ratio of the envelope scale length l to the plasma wavelength λ_p, is invariant under the given transformations, i.e., $\Omega_p L = \Omega_p' L'$. Figures 1–3 show the magnitude $|A_1^{(1)}|$ and phase α of the vector potential, and the scalar potential $\phi_0^{(2)}$ as obtained numerically from the coupled equations (45) and (50) for $\Omega_p = 1$ and an initial condition given by

$$A_1^{(1)}(\xi,\tau=0) = A_0 \mathrm{sech}(A_0\xi/2^{1/2}) , \tag{80}$$

with $A_0 = 1$. The solutions shown should actually be interpreted as being in terms of the primed variables, but the primes have been omitted for simplicity. If we define the initial pulse width L_0 to be the full width at half maximum, then from Eq. (80) we have $L_0 = 3.73/A_0$. Hence, for the values $\Omega_p = A_0 = 1$ used in Figs. 1–3, we have $\Omega_p L_0 = k_p l_0 = 2\pi l_0/\lambda_p = 3.73$, which corresponds to $l_0/\lambda_p \approx 0.6$, i.e., the initial pulse width is smaller than the plasma wavelength. It is evident that the initially symmetric pulse distorts and broadens, with the peak moving to the left while the portion of the pulse in front of the peak continually broadens, extending from the peak into a region of increasing, positive ξ. Since ξ is proportional to the spatial coordinate in the linear group velocity frame, Fig. 1 shows that the peak travels at a velocity less than the linear group velocity. Moreover, as the pulse evolves from the initial condition, it approaches a characteristic shape which has a steep trailing edge and a much broader front portion with a relatively gentle slope. These characteristics can be understood qualitatively by referring to Fig. 3, which shows the potential $\phi_0^{(2)}$. Taking as an example the time $\tau = 32$, Fig. 3 shows that $\phi_0^{(2)}$ is small in the broad front portion of the pulse. Therefore, the nonlinear term $\Omega_p^2 \phi_0^{(2)} A_1^{(1)}$ in Eq. (45) is small so that the right-hand portion of the pulse evolves almost linearly, i.e., it spreads to the right due to the dispersive term $\Omega_p^2 A_{1\xi\xi}^{(1)}$ in Eq. (45). In the region in which the peak and the rear part of the pulse lie, Fig. 3 shows that $\phi_0^{(2)}$ is relatively large, so the nonlinear term in Eq. (45) is apprecia-

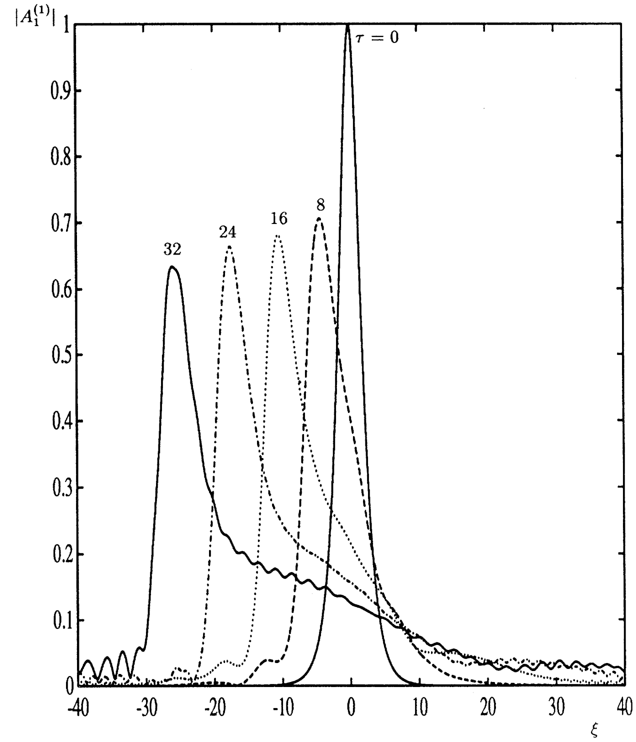

FIG. 1. The magnitude of the lowest-order normalized vector potential for $\Omega_p = 1$ and an initial condition given by $A_1^{(1)}(\xi,\tau=0) = A_0 \mathrm{sech}(A_0\xi/2^{1/2})$ with $A_0 = 1$. In this case, the initial pulse length to plasma wavelength ratio is $l_0/\lambda_p \approx 0.6$. For ease of computation, the solutions in Figs. 1–3 are based on parameters Ω_p and A_0 which result in amplitudes and spatial and temporal scales of order unity (referred to as primed variables in the text). However, Figs. 1–3 are directly applicable to more physically realistic numerical values and scales by scaling the variables according to Eqs. (79) with $\epsilon \ll 1$.

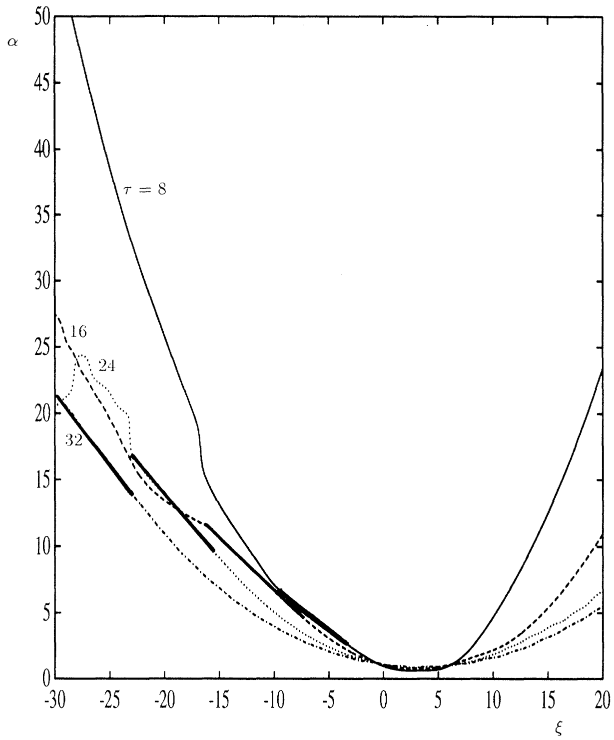

FIG. 2. The phase α of $A_1^{(1)}$ corresponding to the same conditions as in Fig. 1. The almost-linear darker section of each curve indicates the region in which the peak and rear of the pulse lies.

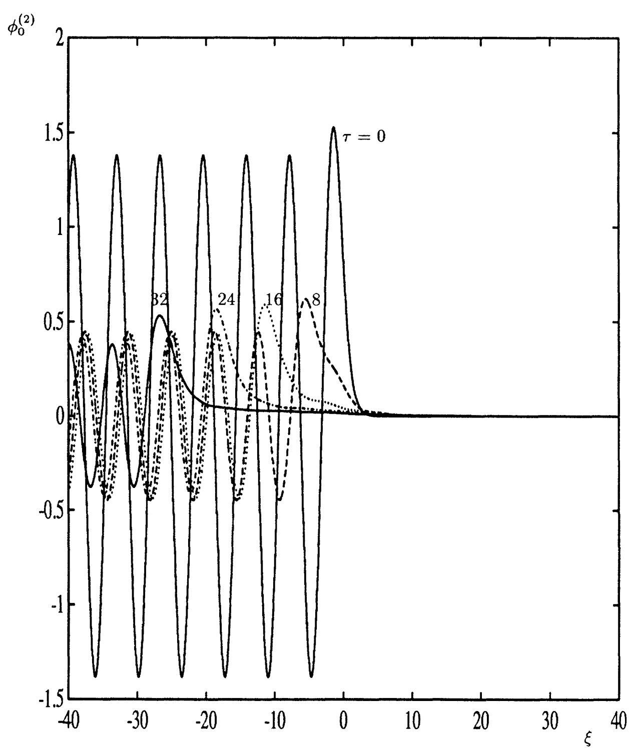

FIG. 3. The lowest-order normalized potential $\phi_0^{(2)}$ corresponding to the same conditions as in Fig. 1.

ble. Letting

$$A_1^{(1)}=\rho(\xi,\tau)\exp[i\alpha(\xi,\tau)]\ , \tag{81}$$

where ρ and α are real, the frequency shift is $\Delta\Omega=-\partial\alpha/\partial T=\epsilon\Omega_0'\partial\alpha/\partial\xi-\epsilon^4\partial\alpha/\partial\tau$, where Eq. (19) has been used. Using Eq. (26), we obtain the lowest-order frequency shift (order ϵ) as $\Delta\Omega\approx\epsilon\partial\alpha/\partial\xi$. Furthermore, the wave-number shift is $\Delta K=\partial\alpha/\partial Z=\epsilon\partial\alpha/\partial\xi$, where Eq. (20) has been used. Thus the lowest order (order ϵ) frequency and wave-number shift are equal and given by

$$\Delta\Omega\approx\Delta K\approx\alpha_\xi\ , \tag{82}$$

where we have let $\epsilon=1$. The equality of the lowest-order frequency and wave-number shift is consistent with the linear dispersion relation (10), which gives $d\Omega=\Omega_0'dK\approx dK$, where the lowest order result $\Omega_0'=1$ from Eq. (26) has been used. Moreover, equal frequency and wave-number shifts produce a group velocity shift which can be calculated from the linear dispersion relation (10) which gives the group velocity $\Omega'=(1-\Omega_p^2/\Omega^2)^{1/2}$. Letting $\Omega=\Omega_0+\Delta\Omega$, where $\Delta\Omega$ is the frequency and wave-number shift, and noting from Eq. (25) that to lowest order $\Omega_0=1$, we obtain $\Omega'\approx1-\Omega_p^2/2+\Omega_p^2\Delta\Omega$. Therefore, the frequency and wave-number shift $\Delta\Omega$ given by Eq. (82) produces a group velocity shift given by

$$\Delta(\Omega')\approx\Omega_p^2\Delta\Omega\approx\Omega_p^2\alpha_\xi\ . \tag{83}$$

Figure 2 shows the phase α of $A_1^{(1)}$ obtained numerically for the same Ω_p and initial condition as in Fig. 1. Since, according to Eq. (82), the frequency and wave-number shift are given by the slope of the curves in Fig. 2, it may be verified from Figs. 1 and 2 that the slope α_ξ is negative in the region in which the peak and the rear part of the pulse lie, which therefore, according to Eq. (83), gives a negative group velocity shift, explaining the leftward movement of the peak and the rear of the pulse in Fig. 1. It is also evident from Figs. 1 and 2 that an appreciable part of the front of the pulse lies in the region in which $\alpha_\xi>0$, which produces a positive group velocity shift, thus explaining the rightward spreading of the front of the pulse in Fig. 1. Moreover, it is evident from Fig. 1 that the portion of the pulse containing the peak and the back of the pulse does not spread appreciably as it moves to the left, in contrast to the front of the pulse which continually spreads to the right with time. This behavior is consistent with Fig. 2, which shows that, at a given time τ, the peak and the rear part of the pulse lie in a region in which the slope α_ξ is approximately constant with respect to ξ. Therefore, all of this part of the pulse moves to the left with a common group velocity. On the other hand, Figs. 1 and 2 show that the broad front of the pulse always lies in a region in which the group velocity shift changes appreciably with ξ, thus explaining the spreading of this part of the pulse.

In summary, the solutions of Eqs. (45) and (50) as shown in Figs. 1–3 indicate that for the initial condition given by Eq. (80), the weakly nonlinear laser pulse continually spreads as it moves through the plasma if the initial pulse width is less than the plasma wavelength.

Three important features of the pulse propagation are the following:

(1) The spreading is confined primarily to the front portion of the pulse and is due essentially to linear dispersion produced by the term $\Omega_p^2 A_{1\xi\xi}^{(1)}$ in Eq. (45). The nonlinear term $\Omega_p^2\phi_0^{(2)}A_1^{(1)}$ s relatively small near the front of the pulse, as shown by a comparison of Figs. 1 and 3. The spreading of the front of the pulse is therefore essentially a linear phenomenon.

(2) The portion of the pulse containing the peak and the rear of the pulse suffers very little spreading and moves with a group velocity less than the linear group velocity. This lack of spreading is due to the nonlinear term $\Omega_p^2\phi_0^{(2)}A_1^{(1)}$ in Eq. (45), which combines with the dispersive term $\Omega_p^2 A_{1\xi\xi}^{(1)}$ to give, at a given time τ, a group velocity shift which is remarkably independent of ξ in the region in which the peak and rear of the pulse are located.

(3) The frequency and wave-number shift vary from a positive value at the front of the pulse to a negative value at the rear. The peak of the pulse has a negative frequency and wave-number shift. The positive frequency and wave-number shift near the front of the pulse is primarily a linear phenomenon, where the negative frequency and wave-number shift near the peak and rear of the pulse is due to a combination of linear and nonlinear effects.

It should be emphasized that these characteristics have been shown to apply to *weakly nonlinear* pulses. For a strongly nonlinear pulse, nonlinear effects may dominate throughout most of the pulse, which can produce a pulse shape and a frequency shift which differ significantly from those found for the weakly nonlinear pulse [17,19]. Moreover, for an initial condition different from Eq. (80), the evolution of a weakly nonlinear pulse may differ from that found here.

As has been noted in Ref. [20], for example, relativistic effects have a profound influence in laser-plasma interactions. Indeed, if the electrons are treated nonrelativistically, it can be shown that an additional nonlinear term $-3\Omega_p^2|A_1^{(1)}|^2A_1^{(1)}/2$ appears on the left-hand side of Eq. (45), which would produces significant changes in the behavior of the solution. Moreover, in the long-pulse case treated in Sec. III, the neglect of relativistic effects causes the coefficient of the nonlinear term in the NLS equation (76) to change from $\Omega_p^2/2$ to $-\Omega_p^2/4$, so that the coefficients of the dispersive and nonlinear terms have opposite signs. In that case, the NLS equation does not have the usual "bright" soliton solution (78) which represents a localized, traveling region in which the envelope intensity is larger than zero, but instead has only "dark" soliton solutions which represent an envelope intensity dip in a continuous-wave background [16,21,22], and can be called an envelope hole. Thus, it is apparent that relativistic effects are of crucial importance in the proper description of a laser pulse in a plasma.

The phase velocity of the wake is of interest in the acceleration of charged particles by the wake field. The solution of Eq. (50) is given by [7]

$$\phi_0^{(2)}(\xi,\tau)=\Omega_p\int_{\xi}^{\infty}d\xi'|A_1^{(1)}(\xi',\tau)|^2\sin[\Omega_p(\xi'-\xi)] \ . \qquad (84)$$

If the entire pulse were to travel with a single velocity, Eq. (84) shows that the wake field would travel at that velocity. Figure 1 shows, however, that different parts of the pulse travel at different velocities. To determine the relationship between the pulse and wake-field velocities, we consider the locations of the maximum value of $|A_1^{(1)}|$, and the first and second maxima of $\phi_0^{(2)}$, denoted by ξ_m, ξ_1, and ξ_2, respectively. Figure 4 shows the velocity $d\xi_m/d\tau$, $d\xi_1/d\tau$, and $d\xi_2/d\tau$ of these maxima obtained numerically for the same parameters and initial condition as in Figs. 1–3. It is evident that, after an initial transient period during which the pulse shape adjusts from the symmetric initial condition to the characteristic asymmetric shape discussed previously, the three velocities are very nearly equal. This indicates that it is a good approximation to assume that the wake-field phase velocity is the same as the velocity of the peak of $|A_1^{(1)}|$. As was discussed previously, the velocity of the peak of $|A_1^{(1)}|$ is modified from the linear group velocity Ω_0' because of the group velocity shift given by Eq. (83). Figure 5 shows the velocity $d\xi_m/d\tau$ for the peak of $A_1^{(1)}$, together with the group velocity shift $\Omega_p^2(\alpha_\xi)_m$, obtained from Eq. (83), where $(\alpha_\xi)_m$ is the nonlinear frequency and wave-number shift evaluated at the location of the peak. The agreement is quite good, which verifies that the negative frequency and wave-number shift are responsible for the backward movement of the peak of the pulse in Fig. 1.

Figures 6 and 7 show the evolution of $|A_1^{(1)}|$ and $\phi_0^{(2)}$ for the case of $l_0/\lambda_p\approx 0.3$, i.e., for a pulse whose initial

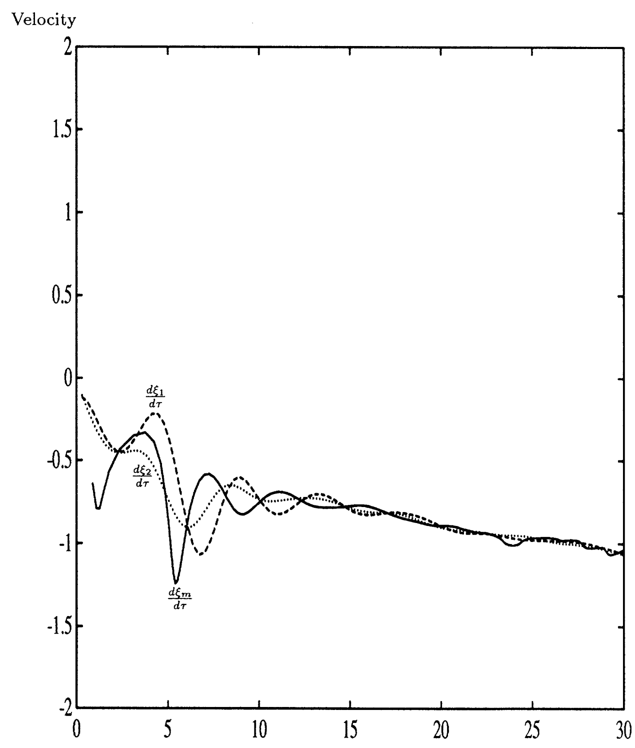

FIG. 4. Velocity $d\xi_m/d\tau$ of the maximum of $|A_1^{(1)}|$, and the velocities $d\xi_1/d\tau$ and $d\xi_2/d\tau$ of the first and second maxima of $\phi_0^{(2)}$ corresponding to the same conditions as in Fig. 1.

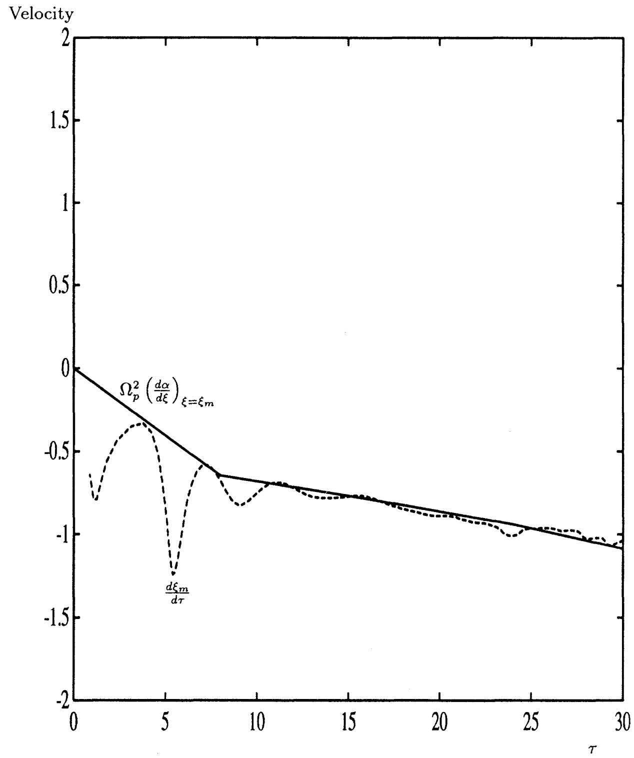

FIG. 5. Group velocity shift $\Omega_p^2(\partial\alpha/\partial\xi)_m$ at the peak of $|A_1^{(1)}|$ compared with the velocity of the peak $d\xi_m/d\tau$ for the same conditions as in Fig. 1.

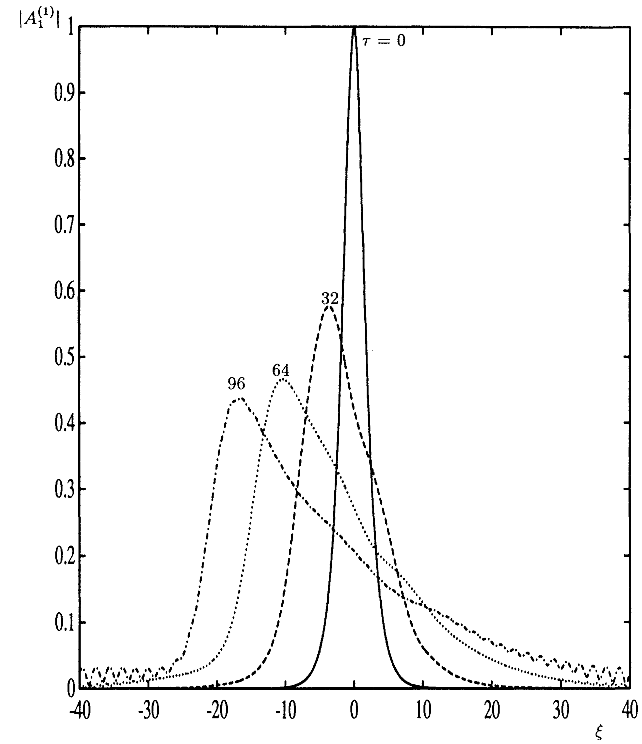

FIG. 6. The magnitude of the lowest-order normalized vector potential for $\Omega_p=0.5$ and an initial condition given by $A_1^{(1)}(\xi,\tau=0)=A_0\mathrm{sech}(A_0\xi/2^{1/2})$ with $A_0=1$. In this case, the initial pulse length to plasma wavelength ratio is $l_0/\lambda_p\approx 0.3$.

width is substantially smaller than a plasma wavelength. It is evident that $|A_1^{(1)}|$ again approaches an asymmetric shape which has the same characteristics found for the longer initial pulse shown in Fig. 1, which include a gradually sloping front portion and a trailing edge with a steeper slope. In this case, however, the time scale required to reach the characteristic shape is larger. Again, the wake-field phase velocity is found to approximately coincide with the velocity of the peak of $|A_1^{(1)}|$.

We have also carried out calculation of $A_1^{(1)}$ and $\phi_0^{(2)}$ for other values of l_0/λ_p and for some other functional forms for the initial pulse (e.g., rectangular). In the cases we have considered, $|A_1^{(1)}|$ approaches a shape which is qualitatively similar to that shown in Figs. 1 and 6 for large τ, provided l_0/λ_p is not large compared with unity.

If the last two terms (the dispersive and nonlinear terms) on the left-hand side of Eq. (45) are of the same order of magnitude, the characteristic time on which the envelope $A_1^{(1)}$ changes is $\tau_e\sim L^2/\Omega_p^2$, which in unnormalized variables is $t_e\sim(k_0 l)^2(\omega/\omega_p)/\omega_p$. The one-dimensional model requires that the laser-beam diffraction time $t_d\sim\pi r_s^2/\lambda c$ be long compared with t_e, where r_s is the laser spot size. This condition is satisfied if

$$r_s/\lambda_p \gg l/\lambda\ . \tag{85}$$

Since the analysis presented here is based on the assumption that $l/\lambda\gg 1$, the condition (85) is more stringent than the condition $r_s/\lambda_p\gg 1$ given in Ref. [7].

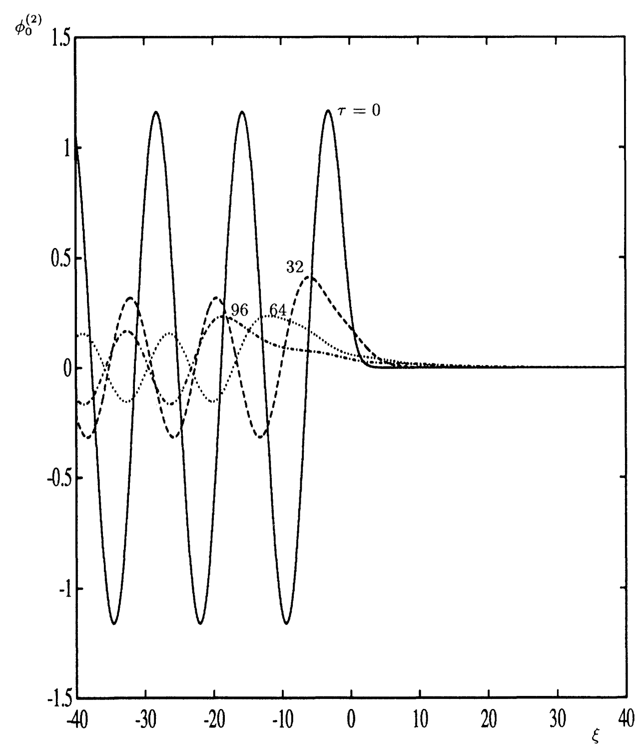

FIG. 7. The lowest-order normalized potential $\phi_0^{(2)}$ corresponding to the same conditions as in Fig. 6.

V. ENERGY VARIATION OF THE LASER PULSE

We calculate the energy in the laser pulse. The normalized energy density is given by

$$w=\tfrac{1}{2}(E^2+B^2+\Omega_p^2 n A^2)\ , \tag{86}$$

where w, E, and B are the energy density, electric field, and magnetic field normalized to (in SI units) $\epsilon_0(k_0c^2m_0/e)^2$, $k_0c^2m_0/e$, and k_0cm_0/e, respectively, and Ω_p, n, and A are the normalized plasma frequency, electron density, and vector potential as defined in Sec. II. We evaluate w through order ϵ^3. Because Ω_p and A are, to lowest order, of order ϵ, the term $\Omega_p^2 n A^2$, due to kinetic energy of the electrons, is of order ϵ^4 and is neglected. Using Eqs. (19), (20), (25), (26), (30), and (31), the electric and magnetic fields through order ϵ^2 are

$$E=-A_T=\epsilon A_\theta^{(1)}+\epsilon^2(A_\theta^{(2)}+A_\xi^{(1)})\ , \tag{87}$$

$$B=A_Z=-E\ . \tag{88}$$

The energy density (86) in this approximation is therefore $w=E^2$. Using Eqs. (30), (31), and (87), we obtain the time-average energy density $\overline{w}$, through order ϵ^3

$$\overline{w}=2[\epsilon^2|A_1^{(1)}|^2+\epsilon^3(A_1^{(1)}A_1^{(2)*}+A_1^{(1)*}A_1^{(2)}) -\epsilon^3 i(A_{1\xi}^{(1)}A_1^{(1)*}-A_{1\xi}^{(1)*}A_1^{(1)})]\ . \tag{89}$$

The laser pulse energy is $W(\tau)=\int_{-\infty}^{\infty}\overline{w}(\xi,\tau)d\xi$, which, upon differentiation with respect to time, yields

$$\frac{dW}{d\tau}=2\int_{-\infty}^{\infty}[(|A_1^{(1)}|^2)_\tau+(A_1^{(1)}A_1^{(2)*}+A_1^{(1)*}A_1^{(2)})_\tau -i(A_{1\xi}^{(1)}A_1^{(1)*}-A_{1\xi}^{(1)*}A_1^{(1)})_\tau]d\xi\ , \tag{90}$$

where, for simplicity, we have let $\epsilon=1$. With the use of the coupled equations for $A_1^{(1)}$ and $\phi_0^{(2)}$, and $A_1^{(2)}$ and $\phi_0^{(3)}$, given in Sec. II, it is shown in Appendix B that Eq. (90) becomes

$$\frac{dW}{d\tau}=\Omega_p^2\int_{-\infty}^{\infty}|A_1^{(1)}|^2\phi_{0\xi}^{(2)}d\xi\ . \tag{91}$$

Using $\Omega_p^2|A_1^{(1)}|^2=\phi_{0\xi\xi}^{(2)}+\Omega_p^2\phi_0^{(2)}$ from Eq. (50), Eq. (91) becomes

$$\frac{dW}{d\tau}=\tfrac{1}{2}\int_{-\infty}^{\infty}[(\phi_{0\xi}^{(2)})^2+\Omega_p^2(\phi_0^{(2)})^2]_\xi d\xi =\tfrac{1}{2}[(\phi_{0\xi}^{(2)})^2+\Omega_p^2(\phi_0^{(2)})^2]|_{-\infty}^{\infty}\ . \tag{92}$$

The contribution from the upper limit vanishes because $\phi_0^{(2)}=\phi_{0\xi}^{(2)}=0$ ahead of the laser pulse. At the lower limit $\phi_0^{(2)}$ satisfies Eq. (50) with $|A_1^{(1)}|^2=0$, whose solution is $\phi_0^{(2)}=\Phi_0\cos[\Omega_p(\xi-\xi_0)]$, where Φ_0 is the amplitude of the wake potential behind the pulse and ξ_0 is a function of τ. Equation (92) therefore becomes

$$\frac{dW}{d\tau}=-\tfrac{1}{2}\Omega_p^2\Phi_0^2\ . \tag{93}$$

Equation (93) is simply a statement of conservation of energy. To lowest order, the energy density in the wake is $(E^2+\Omega_p^2u^2)/2=\Omega_p^2\Phi_0^2/2$, where Eq. (43) and $E=-\phi_{0\xi}^{(2)}$ have been used. To lowest order, the pulse travels with velocity $\Omega_0'\approx 1$. Thus, in a time $d\tau$, the wake energy increases by $\Omega_p^2\Phi_0^2 d\tau/2$, and therefore the pulse energy decreases by that amount, as verified by Eq. (93).

For a short pulse ($\Omega_p L=k_p l\ll 1$), Eq. (84) yields the approximate result

$$\phi_0^{(2)}\approx-\tfrac{1}{2}\Omega_p W\sin[\Omega_p(\xi-\xi_0)]\ , \tag{94}$$

where, from Eq. (89), W is the pulse energy to lowest order given by

$$W=2\int_{-\infty}^{\infty}|A_1^{(1)}|^2 d\xi\ , \tag{95}$$

and ξ_0 is the location of the pulse. Hence, the amplitude of the wake potential is $\Phi_0=\Omega_p W/2$, whereby Eq. (93) becomes $dW/d\tau=-\Omega_p^2W^2/8$, which yields the pulse energy

$$W=(W_0^{-1}+\Omega_p^4\tau/8)^{-1}\ , \tag{96}$$

where W_0 is the pulse energy at $\tau=0$. Thus, at large τ, the pulse energy decreases inversely with time.

Equation (91) can be written in an alternative form by using Eq. (43) to obtain $\phi_0^{(2)}=|A_1^{(1)}|^2-n_0^{(2)}$, whereby Eq. (91) becomes

$$\frac{dW}{d\tau}=\Omega_p^2\int_{-\infty}^{\infty}[(|A_1^{(1)}|^4/2)_\xi-|A_1^{(1)}|^2 n_{0\xi}^{(2)}]d\xi =-\Omega_p^2\int_{-\infty}^{\infty}|A_1^{(1)}|^2 n_{0\xi}^{(2)}d\xi\ . \tag{97}$$

The density perturbation generated by the laser generally has $n_{0\xi}^{(2)}>0$ in the region where $|A_1^{(1)}|^2\neq 0$, so that $dW/d\tau<0$. However, if the laser pulse interacts with the region of a negative density gradient of an externally generated plasma wave, Eq. (97) shows that the pulse energy increases.

VI. FREQUENCY VARIATION OF LASER PULSE

We investigate the relationship between the pulse energy and frequency. Inserting Eq. (81) into Eq. (45), and separating the real and imaginary parts, we obtain

$$\rho\alpha_\tau+\tfrac{1}{2}\Omega_p^2[\rho(\alpha_\xi)^2-\rho_{\xi\xi}-\phi_0^{(2)}\rho]=0\ , \tag{98}$$

$$(\rho^2)_\tau+\Omega_p^2(\rho^2\alpha_\xi)_\xi=0\ . \tag{99}$$

Solving Eq. (98) for $\phi_0^{(2)}$, and inserting the result into Eq.(91), we obtain

$$\frac{dW}{d\tau}=\int_{-\infty}^{\infty}\{2\rho^2\alpha_{\tau\xi}+\Omega_p^2\rho^2[(\alpha_\xi)^2]_\xi-\Omega_p^2\rho^2(\rho_{\xi\xi}/\rho)_\xi\}d\xi\ . \tag{100}$$

The contribution from the last term in the integrand vanishes on integration by parts. From Eq. (99) we obtain

$$\Omega_p^2\rho^2=-(\alpha_\xi)^{-1}\int_{-\infty}^{\xi}(\rho^2)_\tau d\xi\ . \tag{101}$$

Substitution of Eq. (101) into the second term in the integrand of (100) yields

$$\frac{dW}{d\tau}=2\int_{-\infty}^{\infty}\left[\rho^2\alpha_{\tau\xi}-\alpha_{\xi\xi}\int_{-\infty}^{\xi}(\rho^2)_\tau d\xi\right]d\xi\ . \tag{102}$$

Integration of the second term in the integrand in Eq. (102) by parts gives (with $\rho=|A_1^{(1)}|$)

$$\frac{dW}{d\tau}=2\frac{d}{d\tau}\int_{-\infty}^{\infty}|A_1^{(1)}|^2\alpha_\xi d\xi \ . \tag{103}$$

From Eq. (82), the lowest-order (order ϵ) frequency shift is $\Delta\Omega=\alpha_\xi$, which varies over the pulse as shown, for example, in Fig. 2. We define the average frequency shift of the pulse as

$$\overline{\Delta\Omega}=\int_{-\infty}^{\infty}|A_1^{(1)}|^2\Delta\Omega\, d\xi\Big/\int_{-\infty}^{\infty}|A_1^{(1)}|^2 d\xi \ . \tag{104}$$

Noting from Eq. (B2) that the pulse energy to lowest order, as given by Eq. (95), is a constant, and inserting Eq. (104) into (103), we obtain

$$\frac{1}{W}\frac{dW}{d\tau}=\frac{d(\overline{\Delta\Omega})}{d\tau} \ . \tag{105}$$

The average frequency of the pulse is $\overline{\Omega}=\Omega_0+\overline{\Delta\Omega}$, which to order ϵ is $\overline{\Omega}=1+\overline{\Delta\Omega}$, where Eq. (25) has been used. Thus, to lowest order, Eq. (105) becomes $dW/W=d\overline{\Omega}/\overline{\Omega}$, which gives

$$\frac{\overline{\Omega}}{W}=\text{const} \ , \tag{106}$$

which is the well-known relationship between frequency and energy in a closed oscillatory system subjected to a slow adiabatic transformation. Equation (106) gives the result that the average frequency for the laser pulse continually decreases as it loses energy to the wake.

It was shown in Sec. IV that the wake-field phase velocity is approximately equal to the velocity of the peak of $|A_1^{(1)}|$, which in turn travels at the linear group velocity reduced by the modification due to the nonlinear frequency shift. As predicted by Eq. (106), the pulse frequency decreases, so the velocity of the peak of $|A_1^{(1)}|$ and the wake-field phase velocity will also decrease with time.

Using Eqs. (97) and (105), one obtains

$$\frac{d\overline{\Omega}}{d\tau}=-\tfrac{1}{2}\Omega_p^2\int_{-\infty}^{\infty}|A_1^{(1)}|^2 n_{0\xi}^{(2)}d\xi\Big/\int_{-\infty}^{\infty}|A_1^{(1)}|^2 d\xi \ , \tag{107}$$

which relates the frequency variation to the density gradient. For a pulse which is short compared with the scale length of the density gradient, Eq. (107) becomes $d\overline{\Omega}/d\tau\approx-\Omega_p^2 n_{0\xi}^{(2)}/2$, which is equivalent to the result of Ref. [6].

VII. COMPARISON WITH QUASISTATIC APPROXIMATION

Sprangle, Esarey, and Ting [7] have used a quasistatic approximation to study a laser pulse in a plasma. In this approximation, time derivatives are neglected in the plasma equations (4) and (5) after they are transformed to the speed-of-light frame. It is of interest to determine the effect this approximation has on the present analysis, and to compare the approximate and exact results. In the present notation, Eqs. (3c) and (3d) of Ref. [7] in the speed-of-light frame are

$$[n(1-u)]_{\xi'}=n_{\tau'} \ , \tag{108}$$

$$[\gamma(1-u)-\phi]_{\xi'}=-(\gamma u)_{\tau'} \ , \tag{109}$$

where $\xi'=Z-T$ and $\tau'=T$. The relationships between the variables ξ', τ' and those defined in Eqs. (16)–(18) are $\xi=\epsilon\xi'+\epsilon(1-\Omega_0')\tau'$, $\theta=\xi'+(1-\Omega_0)\tau'$, and $\tau=\epsilon^4\tau'$, which give

$$\frac{\partial}{\partial\xi'}=\frac{\partial}{\partial\theta}+\epsilon\frac{\partial}{\partial\xi} \ , \tag{110}$$

$$\frac{\partial}{\partial\tau'}=(1-\Omega_0)\frac{\partial}{\partial\theta}+\epsilon(1-\Omega_0')\frac{\partial}{\partial\xi}+\epsilon^4\frac{\partial}{\partial\tau} \ . \tag{111}$$

If we make the quasistatic approximation [7] by neglecting the right-hand sides of Eqs. (108) and (109), and use Eq. (110), Eqs. (108) and (109) can be written as

$$[n(\Omega_0-u)]_\theta+\epsilon[n(\Omega_0'-u)]_\xi=-(1-\Omega_0)n_\theta-\epsilon(1-\Omega_0')n_\xi \ , \tag{112}$$

$$[\gamma(1-\Omega_0 u)-\phi]_\theta+\epsilon[\gamma(1-\Omega_0' u)-\phi]_\xi=(1-\Omega_0)(\gamma u)_\theta+\epsilon(1-\Omega_0')(\gamma u)_\xi \ . \tag{113}$$

Comparison of Eqs. (23) and (24) with (112) and (113) shows that the quasistatic approximation consists of replacing the right-hand sides of Eqs. (23) and (24) by the right-hand sides of Eqs. (112) and (113), respectively. Noting that the lowest nonvanishing terms in the expansions of n, γu, $1-\Omega_0$, and $1-\Omega_0'$ are of order ϵ^2, it follows that the right-hand sides of Eqs. (23) and (24) vanish at any order less than ϵ^6, whereas the right-hand sides of Eqs. (112) and (113) vanish at any order less than ϵ^4. Therefore, the expansions of n and u based on the quasistatic equations (112) and (113) are expected to differ from those based on Eqs. (23) and (24) beginning at order ϵ^4. Indeed, by carrying out the reductive perturbation expansion based on Eqs. (21), (22), (112), and (113), it is found that $n^{(2)}$, $u^{(2)}$, $n^{(3)}$, and $u^{(3)}$ are unaffected by the quasistatic approximation, but that $n^{(4)}$ and $u^{(4)}$ are changed in that the last term $\Omega_p^2(1-\delta_{0l})n^{(2)}/2$ in both Eqs. (70) and (71) is absent in the quasistatic approximation, which yields the result that the quasistatic approximations of $n_0^{(4)}$, $u_0^{(4)}$, $n_2^{(4)}$, and $u_2^{(4)}$ are in error, but that $u_4^{(4)}$ is exact. Although the equation governing $A_1^{(3)}$ has not been given explicitly, it is obtained from the $l=1$ terms in Eq. (62). Examination of this equation shows that its right-hand side depends on $n^{(4)}$, from which it follows that $A_1^{(3)}$ depends on $n^{(4)}$, which causes $A_1^{(3)}$, found from the quasistatic approximation, to be in error. In addition, it is found that, through order ϵ^5, the quasistatic approximation produces error in $A_1^{(4)}$, $A_1^{(5)}$, $A_3^{(5)}$, $\phi_0^{(4)}$, and $\phi_0^{(5)}$. Thus, if an accuracy of order ϵ^2 in A, and ϵ^3 in n, u, and ϕ is sufficient, then the quasistatic approximation is adequate in predicting the behavior of a weakly nonlinear laser pulse in a plasma. However, if it is desired to have an accurate estimate of a quantity which appears only at higher order in the expansions, such as the third harmonic of the vector potential, which appears first at order ϵ^5, then the quasistatic approximation is inadequate. In fact, $|A_3^{(5)}|$ from the quasistatic approximation is found to be a third of that given by Eq. (64). Thus the power in the third harmonic is larger by a factor of 9 than that derived from the quasistatic approximation.

VIII. SUMMARY AND CONCLUSIONS

We have carried out a perturbation expansion to solve the equations describing a linearly polarized weakly nonlinear laser pulse propagating in plasma in which the electrons are treated relativistically and the ions are assumed stationary. It is assumed that $\omega_p/\omega_0 \ll 1$. The model is one dimensional in that the spatial dependence on the coordinate perpendicular to the direction of propagation is ignored. The use of the reductive perturbation method results in a hierarchy of equations which can be solved to obtain the solution to any desired order in the expansion parameter ϵ. The first level of the hierarchy yields two coupled equations for the envelope of the lowest-order vector potential $A_1^{(1)}$ and the lowest-order scalar potential $\phi_0^{(2)}$. From these equations, it is shown that, for a laser pulse whose spatial extent is long compared with a plasma wavelength, $A_1^{(1)}$ satisfies the NLS equation, which admits closed-form soliton solutions. These long-pulse solitons would probably not be observable experimentally, however, because parametric instabilities occur on a time scale which is shorter than the soliton time scale.

For a pulse which is not long compared with a plasma wavelength, the coupled equations are solved numerically. It is shown that the symmetrical initial pulse given by Eq. (80) broadens and evolves into a characteristic asymmetrical shape which has a gradually sloping front and a relatively steep rear. A frequency and wavenumber shift is produced, which varies spatially over the pulse, being positive near the front of the pulse, but negative over the remainder of the pulse. Pulse broadening is confined primarily to the front portion of the pulse and is due mainly to linear dispersion. The peak and rear part of the pulse travel more slowly than the linear group velocity because of the negative frequency and wavenumber shift which exists in that part of the pulse. Furthermore, the peak and rear part of the pulse are strongly influenced by the plasma nonlinearity, in contrast to the essentially linear behavior of the front portion. It is also shown that the wake-field phase velocity is approximation equal to the velocity of the pulse peak.

An equation governing the pulse energy is derived whose solution shows that the energy of a short pulse decreases inversely proportional to time due to energy transfer to the wake. Moreover, it is shown that, although the frequency shift varies over the pulse, the average pulse frequency is directly proportional to the pulse energy, so that as the pulse loses energy to the wake, the average pulse frequency continually decreases, as well as the phase velocity of the wake field.

The accuracy of the quasistatic approximation, as applied to the weakly nonlinear case, is assessed. It is found that the quasistatic approximation gives results which are accurate through order ϵ^2 in the vector potential A and through order ϵ^3 in the electron density n, longitudinal electron velocity u, and potential ϕ. Since at lowest order A is order ϵ, and n, u, and ϕ are order ϵ^2, the quasistatic approximation gives correct results to one order greater than lowest. Also, it is found that the third-harmonic power is underestimated by the quasistatic theory by a factor of 9.

Since the results derived are based on a one-dimensional model, their validity depends on the transverse dimensions being sufficiently large; the quantitative condition is given by (85). If (85) is not satisfied, transverse diffraction and self-focusing may be important and should be included in the theory.

APPENDIX A: LINEAR SOLUTION

In order to determine the proper scaling of the variables for a weakly nonlinear laser pulse, we consider the linear case. The general solution for any field component f of a linear dispersive mode is

$$f=\int_{-\infty}^{\infty} F(k)\exp\{i[kz-\omega(k)t]\}dk \ , \tag{A1}$$

where $\omega=\omega(k)$ is the dispersion relation. For a pulse of length l with most of the energy in wave numbers close to some value k_0, $F(k)$ is concentrated in the range $\sim l^{-1}$ near k_0, and (A1) may be approximated by $f=\phi\exp[i(k_0z-\omega_0t)]$, where the envelope ϕ is given by

$$\phi=\int_{-\infty}^{\infty} F(k)\exp\{i[(k-k_0)(z-\omega_0't) -\omega_0''(k-k_0)^2t/2]\}dk \ , \tag{A2}$$

and where $\omega_0=\omega(k_0)$, $\omega_0'=\partial\omega(k_0)/\partial k_0$, and $\omega_0''=\partial^2\omega(k_0)/\partial k_0^2$. We assume that the pulse is broad compared with the carrier wavelength, i.e.,

$$k_0 l \gg 1 \ . \tag{A3}$$

We normalize the variables in Eq. (A2) according to

$$K=k/k_0 \ , \quad \Omega=\omega/\omega_0 \ , \quad \kappa=k_0l(K-1) \ , \tag{A4}$$

$$Z=k_0z \ , \quad T=\omega_0t \ , \tag{A5}$$

whereby Eq. (A2) becomes

$$\phi=l^{-1}\int_{-\infty}^{\infty} F\left[k_0\left[1+\frac{\kappa}{k_0l}\right]\right] \times\exp\left[i\left[\frac{\kappa}{k_0l}(Z-\Omega_0'T)-\frac{\kappa^2\Omega_0''T}{2(k_0l)^2}\right]\right]d\kappa \ , \tag{A6}$$

where $\Omega_0'=\partial\Omega(K_0)/\partial K_0=(k_0/\omega_0)\omega_0'$, $\Omega_0''=\partial^2\Omega(K_0)/\partial K_0^2=(k_0^2/\omega_0)\omega_0''$. The function F is appreciable only in the range $|k-k_0|<l^{-1}$ which corresponds to $|\kappa|<1$, i.e., κ is of order unity or less in the region in which the integrand in Eq. (A6) is appreciable. Letting $(k_0l)^{-1}=\epsilon\ll 1$, it is evident from Eq. (A6) that the envelope ϕ is a function of the two variables

$$\epsilon(Z-\Omega_0'T) \ , \quad \epsilon^2\Omega_0''T \ . \tag{A7}$$

In many problems of physical interest, the normalized group dispersion Ω_0'' is of order unity, and therefore the appropriate second variable in (A7) is ϵ^2T [10,11]. However, for a laser pulse in a plasma, the linear dispersion relation is $\omega^2=\omega_p^2+k^2c^2$, which, with the use of Eqs. (A4) becomes

$$\Omega^2=(K^2+\omega_p^2/k_0^2c^2)/(1+\omega_p^2/k_0^2c^2) , \quad \text{(A8)}$$

from which we obtain

$$\Omega_0''=(\omega_p^2/k_0^2c^2)/(1+\omega_p^2/k_0^2c^2)^2 . \quad \text{(A9)}$$

If we assume that $\omega_p \ll k_0 c$, as is usually the case in laser-plasma interactions [7–9], Eq. (A9) gives $\Omega_0''\approx(\omega_p/k_0c)^2\approx(\omega_p/\omega_0)^2\ll 1$. If we assume that $(\omega_p/\omega_0)^2$ is of order ϵ^2, then (A7) shows that the appropriate second variable is $\epsilon^4 T$ for the case of a laser pulse in a plasma if the laser frequency is large compared with the plasma frequency.

APPENDIX B: SIMPLIFICATION OF EQ. (90)

In order to show that Eq. (90) can be reduced to Eq. (91), we multiply Eq. (45) by $A_1^{(1)*}$, and subtract from the result its complex conjugate to obtain

$$i(A_1^{(1)}A_1^{(1)*})_\tau+\tfrac{1}{2}\Omega_p^2(A_1^{(1)*}A_{1\xi}^{(1)}-A_1^{(1)}A_{1\xi}^{(1)*})_\xi=0 . \quad \text{(B1)}$$

Integration of Eq. (B1) yields

$$\int_{-\infty}^{\infty}(|A_1^{(1)}|^2)_\tau d\xi=\frac{d}{d\tau}\int_{-\infty}^{\infty}|A_1^{(1)}|^2 d\xi=0 , \quad \text{(B2)}$$

where it has been assumed that $A_1^{(1)}$ and $A_{1\xi}^{(1)}$ vanish at $\xi=\pm\infty$. Equation (B2) states that $\int_{-\infty}^{\infty}|A_1^{(1)}|^2 d\xi$ is a conserved quantity.

We multiply Eq. (61) by $A_1^{(1)*}$, and subtract from the result its complex conjugate to obtain

$$\begin{aligned} &i(A_1^{(1)*}A_{1\tau}^{(2)}+A_1^{(1)}A_{1\tau}^{(2)*})+\tfrac{1}{2}\Omega_p^2(A_{1\xi\xi}^{(2)}A_1^{(1)*}-A_{1\xi\xi}^{(2)*}A_1^{(1)}) \\ &\quad+\tfrac{1}{2}\Omega_p^2\phi_0^{(2)}(A_1^{(2)}A_1^{(1)*}-A_1^{(2)*}A_1^{(1)}) \\ &\qquad=-A_{1\xi\tau}^{(1)}A_1^{(1)*}+A_{1\xi\tau}^{(1)*}A_1^{(1)} . \end{aligned} \quad \text{(B3)}$$

We multiply Eq. (45) by $A_1^{(2)*}$, and subtract from the result its complex conjugate to obtain

$$\begin{aligned} &i(A_{1\tau}^{(1)}A_1^{(2)*}+A_{1\tau}^{(1)*}A_1^{(2)})+\tfrac{1}{2}\Omega_p^2(A_{1\xi\xi}^{(1)}A_1^{(2)*}-A_{1\xi\xi}^{(1)*}A_1^{(2)}) \\ &\quad+\tfrac{1}{2}\Omega_p^2\phi_0^{(2)}(A_1^{(1)}A_1^{(2)*}-A_1^{(1)*}A_1^{(2)})=0 . \end{aligned} \quad \text{(B4)}$$

The sum of Eqs. (B3) and (B4) can be written as

$$\begin{aligned} &(A_1^{(1)}A_1^{(2)*}+A_1^{(1)*}A_1^{(2)})_\tau \\ &\quad=i\tfrac{1}{2}\Omega_p^2(A_1^{(1)*}A_{1\xi}^{(2)}-A_1^{(2)}A_{1\xi}^{(1)*} \\ &\qquad\quad+A_1^{(2)}A_{1\xi}^{(1)}-A_1^{(1)}A_{1\xi}^{(2)*})_\xi \\ &\quad+i(A_{1\xi\tau}^{(1)}A_1^{(1)*}-A_{1\xi\tau}^{(1)*}A_1^{(1)}) . \end{aligned} \quad \text{(B5)}$$

Substitution of Eqs. (B2) and (B5) into Eq. (90) yields

$$\frac{dW}{d\tau}=2i\int_{-\infty}^{\infty}(A_{1\tau}^{(1)}A_{1\xi}^{(1)*}-A_{1\tau}^{(1)*}A_{1\xi}^{(1)})d\xi , \quad \text{(B6)}$$

where it has been assumed that $A_1^{(1)}$, $A_{1\xi}$, $A_1^{(2)}$, and $A_{1\xi}^{(2)}$ vanish at $\xi=\pm\infty$. Substituting $A_{1\tau}^{(1)}=i(\Omega_p^2/2)(A_{1\xi\xi}^{(1)}+\phi_0^{(2)}A_1^{(1)})$, from Eq. (45), into Eq. (B6), we obtain

$$\frac{dW}{d\tau}=-\Omega_p^2\int_{-\infty}^{\infty}[(A_{1\xi}^{(1)}A_{1\xi}^{(1)*})_\xi+\phi_0^{(2)}(A_1^{(1)}A_1^{(1)*})_\xi]d\xi . \quad \text{(B7)}$$

The first term on the right-hand side integrates to zero. Integrating the second term by parts gives Eq. (91).

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